Notation 0.1. F base field, E/F arbitary extension if not specified otherwise, q quadratic form over F. Moreover, we use the convention for composition, s.t.

$$(a \times b)(c \times d) = \deg(b \cdot c) a \times d$$

which might not be standard.

1 Rost nilpotence (for quadrics) and useful consequences

Theorem 1.1 (Nilpotence thm for quadrics; effective version)**.** *For $d \in \mathbb{Z}_{\geq 0}$ there exists $N(d) \in \mathbb{Z}_{\geq 0}$ s.t. for any d-dimensional quadratic form q over F, the kernel of the restriction*

$$\mathrm{res}_{E|F} : \mathrm{End}(X_q) \to \mathrm{End}(X_{q,E})$$

consists of $N(d)$-power nilpotents for any field extension E/F.

Ideas in the original proof of Rost. Set $X \coloneqq X_q$.

(1): M. Rost's Habilitationsschrift "Chow groups with coefficients" is about generalizing Chow groups, such that in particular with certain coefficients the localisation sequence extends to to a long exact sequence and many other properties of a cohomology theory ...

(2): As in the Serre spectral sequence associated to a Serre fibration, there exists a spectral sequence associated to a fibration (or at least fiber bundle bundle) $F \to E \to B$ over some point $x \in F$:

$$E^2_{p,q} = A_p[B; A_q[F; K^M_*]] \Rightarrow A_{p+q}(E; K^M_*)$$

[1] Applying the spectral sequence to the fiber bundle $X \times B \xrightarrow{pr_2} B$ for $B = X$ and the fact that the spectral sequence is compatible with composition (most likely just by a naturality argument), if we know that $f \in \mathrm{End}(\mathrm{CH}(X_{\kappa(x)}))$ is zero, f also acts trivially on the associated graded of the filtration of $\mathcal{F}A_*(X \times X; K^M_*)$. By inspection of the construction of the filtration for $* = 0$ it is of length $\dim B$ (starting at 0). Hence, $f^{\dim B + 1}$ acts as zero on $\mathrm{Hom}(X, X)$ in particular on Δ_X.

(3): We proceed by induction on d. As we are proving the claim for all field extensions simultaneously, we may assume that q_E is isotropic. As we have seen in the last talk

$$M(X_E) \simeq \mathbb{Z}(0) \oplus M(X_{q_{E,\mathrm{an}}})(1) \oplus \mathbb{Z}(\dim X - 1)$$

[1] K^M_* is the Milnor K-theory.

Now this decomposition gives rise to a matrix representation of f. This matrix is a triangular matrix as

$$\begin{matrix} \mathrm{Hom}(\mathbb{Z}(0), M(X_{q_{E,\mathrm{an}}})(1)) = 0 & \mathrm{Hom}(\mathbb{Z}(0), \mathbb{Z}(\dim X + 1)) = 0 \\ & \mathrm{Hom}(M(X_{q_{E,\mathrm{an}}})(1), \mathbb{Z}(\dim X + 1)) \end{matrix}$$

for dimension reasons. Moreover, $\mathrm{End}(\mathbb{Z}(i)) \to \mathrm{End}(\mathbb{Z}(i)_E)$ is an isomorphism, the two outer diagonal entries are 0. By induction the middle diagonal entry is $N(d-1)$-power nilpotent. Then ② shows that $N(d) := N(d-1)(d+1)$ does the job. □

As we will see the condition above proves to be useful in studying motivic decompositions. Hence, we give it a name.

Definition 1.2 ((Rost) nilpotence/nilpotence principal)**.** A motive M satisfies Rost nilpotence/the nilpotence principal if and only if for every field extension E/F

$$\mathrm{res}_{E|F} : \mathrm{End}(M) \to \mathrm{End}(M_E)$$

has kernel consisting of nilpotents.

Remark 1.3. If $X \in \mathrm{Sm}(F)$ satisfies Rost nilpotence and $\pi \in \mathrm{End}(X)$ is a projector, then (X, π) satisfies Rost nilpotence as

$$\begin{array}{ccc} \mathrm{End}(X) & \xrightarrow{\mathrm{res}_{E|F}} & \mathrm{End}(X_E) \\ \cup\!\shortmid & & \cup\!\shortmid \\ \pi\mathrm{End}(X)\pi & \longrightarrow & \pi_E\mathrm{End}(X)\pi_E \end{array}$$

where $\pi_E := \mathrm{res}_{E|F}(\pi)$.

For the rest of this section let M, N denote motives satisfying Rost nilpotence.

Definition 1.4. We call a cycle $\alpha \in \mathrm{CH}(V_E)$ F-rational if $\alpha \in \mathrm{im}(\mathrm{res}_{E|F})$.

Let M be an arbitrary motive

Corollary 1.5. *Given $p \in \mathrm{End}(M_E)$ an F-rational projector, we find an idempotent lift. Moreover, if $M = M(X)$ for some variety X and given F-rational pairwise orthogonal projectors $\rho_1, \ldots, \rho_k \in \mathrm{End}(M_E)$ with $\sum \rho_i = \Delta_{M_E}$ constitute a decomposition*

$$M = \bigoplus_{i=1,\ldots,k} (X, \rho_i)$$

Proof. Using the theorem above this boils down to idempotent lifting, a theorem from basic commutative algebra. The addendum is obtained by using that given a ring map $A \xrightarrow{f} B$ (not necessarily between commutative) rings with kernel consisting of nilpotents and $a \mapsto b$ projectors gives rise to the decomposition

$$A \simeq A/(1-a)A \times A/a \xrightarrow{\bar{f}_1 \times \bar{f}_2} B/(1-b) \times B/b \simeq B$$

allowing for induction on k. □

Corollary 1.6. *Let $f \in \mathrm{Hom}(M,N), g \in \mathrm{Hom}(N,M)$ such that $f_E := \mathrm{res}_{E|F}(f)$ is an isomorphism. Then f is an isomorphism, too.*

Proof. This with the commutative algebra fact that for $A \to B$ a finite ring map with kernel consisting of nilpotents, then if a, a' become inverses upon applying f, both are units: By assumption $aa' - 1$ is a nilpotent, hence, aa' and by the same argument $a'a$ are units and therefore a, a' are. □

Remark 1.7. If we assume that M, N are motives associated to quadrics (or any varieties whose motives split into a finite sum of Tates over some field extension), then we need not assume the existence of g.

Remark 1.8. So in many cases we can reduce the question whether some motive has a certain decomposition to the question whether some cycle is F-rational. For this we observe what our knowledge of Chow groups tells us about F-rationality:

$$\mathrm{CH}(\mathbb{P}^n_F) \xrightarrow[\simeq]{\mathrm{res}_{E|F}} \mathrm{CH}(\mathbb{P}^n_E)$$
$$\mathrm{CH}(X_q) \xrightarrow[\simeq]{\mathrm{res}_{E|F}} \mathrm{CH}(X_{q,E}) \text{ , } q \text{ split quadratic form}$$

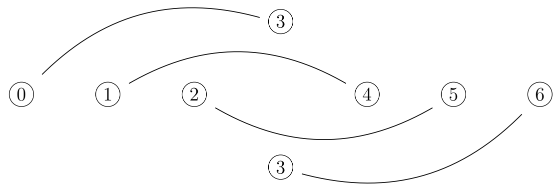
and, since in both cases we have the Künneth formula available, the same holds for products with themselves.

2 The motive of a Pfister quadric

Theorem 2.1. *Let π an n-fold anisotropic Pfister form (or a scalar multiple)*

$$M(X_\pi) \simeq \bigoplus_{i=0,\dots,2^{n-1}-1} R_\pi(i)$$

Example 2.2. In the case we have a 3-fold, hence, 8-dimensional Pfister form. So the picture is the following:



Remark 2.3. Let $\pi : V \to F$ a split $\geq$ 2-fold Pfister form, $X \coloneqq X_\pi$, with maximal totally isotropic subspace $W \subseteq V$. Then the image of $\mathrm{CH}(\mathbb{P}(W)) \to \mathrm{CH}(X_\pi)$ and the image of a hyperplane h by the pullback map $\mathrm{CH}(\mathbb{P}(V)) \to \mathrm{CH}(X_\pi)$, which we also denote by h generate $\mathrm{CH}(X_\pi)$. More specifically, $h_W^i \mapsto l_{\dim W-1-i}$ and h^k $k = 0, ..., \dim W$ form a $\mathbb{Z}$-basis. Moreover we have the multplicative relations

1. $hl_i = l_{i-1}$
2. $h^{\frac{\dim X_q}{2}+1} = 2l_{\frac{\dim X_q}{2}-1}$
3. $l_d l'_d = l_0$ and from this we deduce $l_d^2 = 0 = l'^2_d$

Observe that h is F-rational even if π is not split. In fact, l_i is F-rational iff $i(q) > i$ by result from last talk. Moreover, observe that $2l_d$ is F-rational in our specific case: Fix L/F a quadratic field extension, giving rise to the following diagram

$$\begin{array}{ccc} \mathrm{CH}_d(X_L) & \xrightarrow{\mathrm{corestr}_{L|F}} & \mathrm{CH}_d(X) \\ \downarrow{\scriptstyle \mathrm{res}_{L(\pi)|L}} & & \downarrow{\scriptstyle \mathrm{res}_{F(\pi)|F}} \\ \mathrm{CH}_d(X_{L(\pi)}) & \xrightarrow{\mathrm{corestr}_{L(\pi)|F(\pi)}} & \mathrm{CH}_d(X_{F(\pi)}) \end{array}$$

where the corestriction of a finite field extension E/F is just the pushforward along the map $X_E \to X$ (proper, as $E \to F$ is proper iff $E \to F$ is of finite type iff $E \to F$ is finite (last equivalence is a consequence of Noether normalisation)). The square commutes as applying $X \times \_$ to the fiber square

$$\begin{array}{ccc} L(\pi) & \longrightarrow & L \\ \downarrow & & \downarrow \\ F(\pi) & \longrightarrow & F \end{array}$$

yields a fiber square and then using the push-pull formula. Now we chase $l_d \in \mathrm{CH}(X_L)$ through the diagram:

$$\begin{array}{ccc} l_d & \longmapsto & l' \\ \downarrow & & \downarrow \\ l_d & \longmapsto & [L(\pi):F(\pi)]\, l_d = 2l_d \end{array}$$

where in the last line we use that the linear subspace $\mathbb{P}(W_{L(\pi)})$ gets mapped to $\mathbb{P}(W_{F(\pi)})$ or alternatively the restriction-corestriction formula[2].

Proof of Thm. 2.1. We will only prove the theorem for ≥ 2-fold Pfister form as then $\dim X_q \equiv 2 \mod 4$. In the two dimensional case the statement is just that the motive of X_π does not split into Tates over the basefield as its anisotropic.

Set $X \coloneqq X_\pi$ and $2d = \dim X$. Let $E = F(\pi)$ (so that we know that $2l_d$ over E is F-rational by remark 2.3) and set $\bar{X} \coloneqq X_E$. As already mentioned, the constructing the decomposition a la Karpenko

1. Find a decomposition of $\Delta_{\bar{X}}$ into orthogonal projectors
2. and show that they are F-rational
3. and show that they are Tate twists of one another

For (1): As we have seen last talk

$$\Delta_{\bar{X}} = [\sum_{i=0}^{d-1} l_k \times h^k + h^k \times l^k] + l_d \times l'_d + l'_d \times l_d = [\sum_{i=0}^{d-1} l_k \times h^k + h^k \times l^k] + l_d \times (l'_d - l_d) + (l_d + l'_d) \times l_d$$

for $l'_d \coloneqq h^d - l_d$. Our candidate for pairwise orthogonal idempotents are

$$\pi_k \coloneqq h^k \times l_k + l_{d-k} \times h^{d-k}$$

for $k = 1, ..., d-1$ and

$$\pi_0 \coloneqq 1 \times l_0 + l_d \times (l'_d - l_d)\ ,\ \pi_d \coloneqq l_0 \times 1 + (l'_d + l_d) \times l_d$$

[3]

For (2): Our intermediate goal: Show that $1 \times l_d + l_d \times 1$ is F-rational.

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[2] The restriction-corestriction formula says that $\mathrm{corestr}_{L|F} \circ \mathrm{res}_{L|F} = [L:F]$. One might recall a similar formular for Galois cohomology from number theory

[3] Checking pairwise orthogonal and idempotent for these guys is just a verification.

Observe that $\mathrm{Spec}F(X) \xrightarrow{\nu} X$ is a flat morphism of relative dimension $-\dim X$ (on affines its a localisation), hence, we have a pullback morphism

$$\mathrm{CH}_{3d}(X \times X) \to \mathrm{CH}_d(X_{F(q)})$$

This morphism is easily seen to be surjective[4]

Now consider the following diagram

$$\begin{array}{ccc} \mathrm{CH}_{3d}(X \times X) & \twoheadrightarrow & \mathrm{CH}_d(X_{F(q)}) \\ \downarrow{\scriptstyle \mathrm{res}_{E|F}} & & \simeq \downarrow{\scriptstyle \mathrm{res}_{E(q)|F(q)}} \\ \mathrm{CH}_{3d}(\bar{X} \times \bar{X}) & \longrightarrow & \mathrm{CH}_d(\bar{X}_{E(q)}) \end{array}$$

Pick a lift α of l_d along the top map. Then by Künneth

$$\mathrm{res}_{E|F}(\alpha) = l_d \times 1 + a1 \times l_d + \sum_{k=0}^{d} a_i h^k \times h^{d-k}$$

for some $a_i, a \in \mathbb{Z}$, since for dimension reasons and inspection of the definition of pullbacks

$$f^*(l_i) = 0 \; , \; f^*(h^i) = \begin{cases} 0 & , \text{ if } i \neq 0 \\ 1 & , \text{ else} \end{cases}$$

As h is defined over F and $2l_d$ is defined over F by restriction-corestriction argument in remark 2.3, $l_d \times 1$ or $\rho := l_d \times 1 + 1 \times l_d$ is F-rational. But in the first case, observe that $1 \times l_d = (l_d \times 1)^t$ such that ρ is again F-rational.

Multiplying with $h^i \times h^{d-i}$, we obtain the non-special projectors in (1) and observing that

$$\begin{aligned} 1 \times l_0 + l_d \times h^d &= (1 \times h^d)(1 \times l_d + l_d \times 1) \\ l_0 \times 1 + h^d \times l_d &= (h^d \times 1)(1 \times l_d + l_d \times 1) \end{aligned}$$

and using that 2 times a cycle on the product $\bar{X} \times \bar{X}$ by the same argument as in remark 2.3, we obtain that

$$\begin{aligned} \pi_0 &= 1 \times l_0 + l_d \times h^d - 2l_d \times l_d \\ \pi_d &= l_0 \times 1 + h^d \times l_d \end{aligned}$$

[4]It suffices to show that the map is a surjection on generators: Let V be a variety in $X \times F(\pi)$, then its generic point ν_V lies in some affine $U = \mathrm{Spec}A$. Then $\mathrm{Spec}(A \otimes \mathrm{Frac}(A)) \simeq U \times F(\pi) \to U \times U \simeq \mathrm{Spec}(A \otimes A)$ is a localisation, hence,

$$Z_k(U \times U) \to Z_k(U \times F(\pi))$$

has $[V]$ in its image. Also $U \times U \hookrightarrow X \times X$ induces a surjection on cycles via pullback, hence, the statement follows.

are rational.

For ③: $M(X)$ satisfies Rost nilpotence and $\mathrm{Hom}((X_E, \pi_k), (X_E, \pi_l)) = 0$, i.e., over E all endomorphisms of $M(X)$ are in diagonal form. Hence, (X, π_k) must also satisfy Rost nilpotence. Moreover, over E an easy computation (with the degree formula from the talk before last talk) shows that

$$(\bar{X}, \pi_0)(k) \underset{h^k \times l_0 + l_{d-k} \times h^d}{\overset{1 \times l_k + l_d \times h^{d-k}}{\rightleftarrows}} (\bar{X}, \pi_k)$$

are inverse equivalences for $k < d$ for $k = d$ one has to take

$$(\bar{X}, \pi_0)(d) \underset{(l'_d + l_d) \times \times l_0 + l_0 \times (l'_d - l_d) =: \beta}{\overset{1 \times l_d + l_d \times 1}{\rightleftarrows}} (\bar{X}, \pi_d)$$

We can write

$$\begin{aligned} 1 \times l_k + l_0 \times h^{d-k} &= 1 \times h^{d-k}(1 \times l_d + l_d \times 1) \\ h^k \times l_0 + l_{d-k} \times h^d &= h^k \times h^d(1 \times l_d + l_d \times 1) \end{aligned}$$

For the last cycle we need to show is F-rational, first we observe that

$$h^d \times l_0 + l_0 \times h^d = (h^d \times h^d)(1 \times l_d + l_d \times 1)$$

is rational. Subtracting $2l_0 \times l_d$, which is F-rational by a similar restriction-corestriction argument as in remark 2.3, yields β; hence β is rational, too. So applying Corollary 1.6 finishes the proof. □

Remark 2.4. Using Corollary 1.6 one shows that R_π is uniquely determined as the first motivic summand of $M(X_\pi)$ (see Prop 6.4 in "Minimal Pfister neighbors via Rost projectors" or in Rost's original paper).

Proposition 2.5. *Given a decomposition $\phi \perp \phi' = \pi$, where ϕ is a Pfister neighbor of π. Then*

$$M(X_\phi) = \bigoplus_{i=0,\ldots,m-1} R_\pi(i) \oplus M(X_{\phi'})(m)$$

where $m = \frac{\dim\phi - \dim\phi'}{2}$.

Proof sketch. This theorem is stated without proof in a paper by Karpenko and Merkurjew, where it is said that similar methods to the ones in the proof above can be applied. We are going to go a different route, using methods from EKM, namely the following strong theorem:

Claim (Vishik). Let X, Y be quadrics. Then

$$M(X) \simeq M(Y) \Leftrightarrow \dim X = \dim Y \text{ and } i(X_L) = i(Y_L) \text{ for all fields } L/F$$

Moreover, recall that $F(\pi)/F$ can be written as a purely transcendental extension K/F followed by a quadratic extension $F(\pi) = M/K$.

Applied to motives, the following claim says that purely transcendental extensions allow easy lifting of motivic decompositions.

Claim. Let K/F be purely transcendental. Then for any $X \in \mathrm{Sm}_F$ $\mathrm{res}_{K|F} : \mathrm{CH}(X) \to \mathrm{CH}(X_K)$ is surjective.

Proof. Note that we can write $K = F(\mathbb{A}^n_F)$ for some $n \in \mathbb{N}$. By functoriality of pullbacks we can write $\mathrm{res}_{K|F}$ as

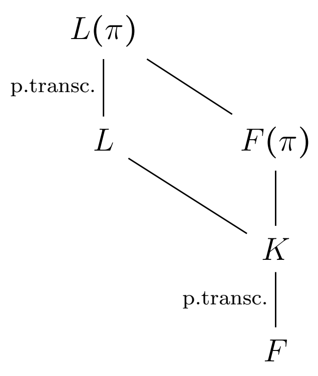
$$\mathrm{CH}^*(X) \to \mathrm{CH}^*(X \times \mathbb{A}^n_F) \to \mathrm{CH}^*(X \times \mathrm{Spec}(K))$$

By $\mathbb{A}^1$-invariance the first map is an isomorphism and by a previous argument the second map is surjective. □

So by Witt theory of quadratic field extensions we have

$$\phi \simeq q \perp q'\langle 1, -a\rangle$$

over K with $\dim q' = m$ s.t. $q_M \simeq \phi'_M$. Now we want to apply Vishik's theorem (for integral coefficients): By the claim above (up to identifying the Tate motives, which is relatively easy) we only need to apply Vishik's result with K as our base field. Over K both are anisotropic as they are isomorphic and anisotropic over $F(\pi)$. So assume L/K is a field extension making either q or q' isotropic. Then π_L is split, hence, we have a diagram as follows



As purely transcendental extensions do not change Witt indices and $q_{F(\pi)} \simeq \phi'_{F(\pi)}$, we are done.

□

Remark 2.6 (Motivic decomposition of an excellent form)**.** Let q be an excellent form, i.e.,

$$q = \sum_{i=0}^{r} (-1)^i \pi_i$$

in $W(F)$ and $\pi_0 \supset \pi_1 \supset \ldots \supset \pi_r$ are anisotropic Pfister forms such that $2 \dim \pi_r < \dim \pi_{r-1}$.

Observe that $\pi_0 = q \perp q'$, where q' is the excellent anisotropic form represented by $\sum_{i=1}^{r} (-1)^{i+1} \pi_i$ (whose motivic decomposition we know by induction). Now Proposition 2.5 allows us to compute the motive of X_q in terms of $X_{q'}$

In the case $q = 11\langle 1 \rangle$, $11 = 2^4 - 2^3 + 2^2 - 2^0$, we therefore get the following motivic decomposition: